On the Solutions of Linear Non-homogeneous Partial Differential Equations.

By L. L. Steimley.

The integrals of a partial differential equation of the first order were first classified by Lagrange, who separated them into three groups; namely, the general, the complete, and the singular integrals. For a long time this classification was thought to be complete. In fact, Forsyth, in his "Differential Equations," published first in 1885, gives a supposed proof of a theorem stating that every solution of such a differential equation is included in one or other of the three classes named. This error is also carried through the second and third English editions and the two German editions, the last one being published in 1912.

In 1891 Goursat pointed out, in his "Équations aux dérivées partielles du premier ordre," that solutions exist which do not belong to any of these three classes, and showed, indeed, that the existing theory is not complete even for the simplest forms.

In November, 1906, Forsyth, in his presidential address to the London Mathematical Society, emphasized the fact that the theory is incomplete, and in his closing remark said: "It appears to me that there is a very definite need for a re-examination and a revision of the accepted classification of integrals of equations even of the first order: in the usual establishment of the familiar results, too much attention is paid to unspecified form, and too little attention is paid to organic character, alike of the equations and of the integrals. Also, it appears to me possible that, at least for some classes of equations, these special integrals may emerge as degenerate forms of some semi-general kinds of integrals; but it is even more important that methods should be devised for the discovery of these elusive special integrals." Forsyth also, in an address delivered at the Fourth International Congress of Mathematicians, takes advantage of the opportunity offered, to again emphasize the incompleteness of the existing theory of partial differential equations of the first order.

Some indication of the nature of the incompleteness can be seen from the equation

$$xp+yq=z.$$

A general solution by Lagrange's method is afforded by

$$f(\phi_1, \phi_2)=0,$$

where f is an arbitrary function and where $\phi_1=\frac{z}{x}$ and $\phi_2=\frac{z}{y}$. We can easily verify that

$$\psi=z-\frac{x^2}{y}=0$$

affords a solution of the original equation; but this solution cannot be put in the form of the Lagrange general integral, since the Jacobian

$$\frac{\partial(\phi_1, \phi_2, \psi)}{\partial(z, x, y)}$$

is not identically zero. Consequently the so-called general solution is not a general solution in the true sense of the term, since not every solution of the differential equation can be put in that form.

In the present paper we shall deal with the linear non-homogeneous equation

$$\sum_{i=1}^{n} X_i p_i=Z, \qquad p_i=\frac{\partial z}{\partial x_i}, \tag{1}$$

where X_i and Z are functions of the variables $z, x_1, x_2, \ldots, x_n$. We shall assume, as we may without loss of generality, that all common factors of Z, $X_1, X_2, \ldots, X_n$ have been removed; and consequently we need not take into account the values of z which satisfy simultaneously

$$Z=0, \quad X_i=0, \qquad i=1, 2, \ldots, n.$$

We also assume that there is a set of values of the variables $z, x_1, x_2, \ldots, x_n$, in the vicinity of which the functions X_i and Z are single-valued and analytic. Throughout the paper we shall confine our attention to such a domain.

In this paper a new and complete classification is given of all the integrals of equation (1). In the new classification, the so-called special integrals (except for trivial ones) are contained in the general solution. A means is developed by which all these elusive special integrals can be readily determined as soon as the Lagrange general integral is known.

In order to obtain the solution of (1) we take the set of equations

$$\frac{dx_i}{X_i}=\frac{dz}{Z}, \qquad i=1, 2, \ldots, n. \tag{2}$$

It is well known that (2) possesses n functionally independent integrals, one or more of which contain z explicitly. Let these integrals be

$$\phi_i(z, x_1, x_2, \ldots, x_n)=c_i, \qquad i=1, 2, \ldots, n, \tag{3}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. Not all the above n equations are free of z.

Any $\phi_i(z, x_1, x_2, \ldots, x_n)=c_i$ affords a solution of the original equation (1) if it involves z explicitly. Since it is an integral of the equations (2), the relations

$$\sum_{r=1}^{n} \frac{\partial \phi_i}{\partial x_r} dx_r + \frac{\partial \phi_i}{\partial z} dz=0, \qquad i=1, 2, \ldots, n, \tag{4}$$

are consistent with (2). From (2) and (4) follows immediately

$$\sum_{r=1}^{n} \frac{\partial \phi_i}{\partial x_r} X_r + \frac{\partial \phi_i}{\partial z} Z=0, \qquad i=1, 2, \ldots, n. \tag{5}$$

If the equation

$$f(\phi_1, \phi_2, \ldots, \phi_n)=0,$$

in which f is an arbitrary function of the quantities $\phi_1, \phi_2, \ldots, \phi_n$, determines one or more values of z, then it affords solutions of (1), and is what may be called the Lagrange general integral.

We have already seen that not all the solutions of every equation (1) can be put in this form, and since they cannot, we wish to see just how they differ from it. We also wish to get a solution which will be general in the true sense of the term.

Let

$$\psi(z, x_1, x_2, \ldots, x_n)=0 \tag{6}$$

be any equation whatever with the restrictions that it determines a z which satisfies equation (1) and that there is a set of values of the variables $z, x_1, x_2, \ldots, x_n$, in the domain which we are considering, such that in their vicinity ψ and z are single-valued and analytic functions.

Since $\phi_1, \phi_2, \ldots, \phi_n$ are functionally independent, not all the n-th order determinants of the matrix

$$\left\| \begin{matrix} \frac{\partial \phi_1}{\partial z} & \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial z} & \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi_n}{\partial z} & \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{matrix} \right\|$$

vanish identically.

Consequently one of the Jacobians

$$J=\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

or

$$J_s=\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)}, \qquad s=1, 2, \ldots, n,$$

does not vanish identically.

If $J \neq 0$, then we can solve the system $\phi_i = \phi_i(z, x_1, x_2, \ldots, x_n)$, $i=1, 2, \ldots, n$, for $x_1, x_2, \ldots, x_n$, in terms of $z, \phi_1, \phi_2, \ldots, \phi_n$. Substituting these values in equation (6), we get

$$\psi(z, x_1, x_2, \ldots, x_n) = g(z, \phi_1, \phi_2, \ldots, \phi_n) = 0. \tag{6'}$$

Two cases now arise. Either z is expressible in terms of $\phi_1, \phi_2, \ldots, \phi_n$, or equation (6′) may be written in the form

$$\theta(z) g_2(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

where $\theta(z)$ is independent of $\phi_1, \phi_2, \ldots, \phi_n$. $\theta(z)$ may be a constant.

If z is expressible in terms of $\phi_1, \phi_2, \ldots, \phi_n$, let us write

$$z = g_1(\phi_1, \phi_2, \ldots, \phi_n). \tag{7}$$

Differentiating z with respect to x_i $(i=1, 2, \ldots, n)$ we get

$$\frac{\partial z}{\partial x_i} = p_i = \sum_{r=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_i} + \frac{\partial \phi_r}{\partial z} p_i\right), \qquad i=1, 2, \ldots, n. \tag{8}$$

Multiplying the i-th equation in (8) by X_i and adding the equations thus obtained, we get

$$\sum_{i=1}^{n} X_i p_i = \sum_{i=1}^{n} X_i \sum_{r=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_i} + \frac{\partial \phi_r}{\partial z} p_i\right) = \sum_{r=1}^{n}\sum_{i=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_i} X_i + \frac{\partial \phi_r}{\partial z} X_i p_i\right).$$

Hence, employing (5) and (1), we have

$$\sum_{i=1}^{n} X_i p_i = \sum_{r=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(-\frac{\partial \phi_r}{\partial z} Z + \frac{\partial \phi_r}{\partial z} Z\right),$$

or

$$Z = 0. \tag{9}$$

That is, if

$$z = g_1(\phi_1, \phi_2, \ldots, \phi_n)$$

is a solution of equation (1), then this value of z satisfies $Z=0$.

Consequently, if we wish to get solutions of form (7), we need to examine only those values of z which are determined by equation (9) and see whether they satisfy equation (1). Since no knowledge of differential equations or integration is involved in getting these solutions, it follows that they are trivial.

If z is not a function of $\phi_1, \phi_2, \ldots, \phi_n$, then equation (6′) can be written in the form

$$\theta(z) g_2(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

where $\theta(z)$ is a function of z alone and may be a constant.

Since the values of z determined by $\theta(z)=0$ afford solutions of equation (1), we can easily determine the character of θ. The values of z determined by $\theta(z)=0$ are of the form $z=$ a constant. (By exception this constant may

be infinity.) If $z=\alpha$, a finite constant, is a solution of equation (1), a direct substitution in that equation shows that for $z=\alpha$ we have

$$Z\equiv 0$$

for all values of $x_1, x_2, x_3, \ldots, x_n$.

Therefore $\theta(z)=0$ determines only those (non-infinite) values of z which satisfy $Z=0$.

If

$$J_s=\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)}\not\equiv 0,$$

where s is one of the integers $1, 2, \ldots, n$, then the system $\phi_i=\phi_i(z, x_1, x_2, \ldots, x_n)$, $i=1, 2, \ldots, n$, can be solved for $z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n$ in terms of $x_s, \phi_1, \phi_2, \ldots, \phi_n$.

On substituting these values of z and x_i (x_s excluded) in (6), we get

$$\psi(z, x_1, x_2, \ldots, x_n)=h(x_s, \phi_1, \phi_2, \ldots, \phi_n)=0. \tag{6''}$$

Again two cases arise. Either x_s is expressible in terms of $\phi_1, \phi_2, \ldots, \phi_n$, or

$$h(x_s, \phi_1, \phi_2, \ldots, \phi_n)\equiv\theta_s(x_s)h_2(\phi_1, \phi_2, \ldots, \phi_n)=0,$$

where $\theta_s(x_s)$ is a function of x_s alone and may be a constant.

If x_s is expressible as a function of $\phi_1, \phi_2, \ldots, \phi_n$, let it be written as

$$x_s=h_1(\phi_1, \phi_2, \ldots, \phi_n). \tag{10}$$

Differentiating x_s with respect to x_i $(i=1, 2, \ldots, n)$, we get

$$\delta_{is}=\sum_{r=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_i}+\frac{\partial\phi_r}{\partial z}p_i\right), \qquad i=1, 2, \ldots, n, \tag{11}$$

where δ_{is} is equal to 1 or 0 according as i is or is not equal to s.

Multiplying equations (11) in order by $X_1, X_2, \ldots, X_n$ and adding, we get

$$X_s=\sum_{i=1}^{n}X_i\sum_{r=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_i}+\frac{\partial\phi_r}{\partial z}p_i\right)=\sum_{r=1}^{n}\sum_{i=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_i}X_i+\frac{\partial\phi_r}{\partial z}X_ip_i\right).$$

Hence, employing (5) and (1), we get

$$X_s=\sum_{r=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(-\frac{\partial\phi_r}{\partial z}Z+\frac{\partial\phi_r}{\partial z}Z\right),$$

or

$$X_s=0. \tag{12}$$

That is, if

$$x_s=h_1(\phi_1, \phi_2, \ldots, \phi_n)$$

affords a solution of (1), then this value of x_s also satisfies (12).

Consequently we have the trivial case again. Such solutions can be determined by processes independent of integration and involve no knowledge of

partial differential equations. To get these solutions we take those values of z which cause X_s to vanish, and determine whether they satisfy equation (1).

If x_s is not a function of $\phi_1, \phi_2, \dots, \phi_n$, then equation (6″) assumes the form

$$\theta_s(x_s)h_2(\phi_1, \phi_2, \dots, \phi_n)=0,$$

where $\theta_s(x_s)$ is a function of x_s alone and may be a constant.

Summing up our results we have the following theorem:

Let the general linear non-homogeneous partial differential equation be written in the form

$$\sum_{i=1}^{n} X_i(z, x_1, x_2, \dots, x_n)p_i=Z(z, x_1, x_2, \dots, x_n), \qquad p_i=\frac{\partial z}{\partial x_i}. \tag{A}$$

We assume that there is a set of values of the variables $z, x_1, x_2, \dots, x_n$, *in the vicinity of which the functions* $Z, X_1, X_2, \dots, X_n$ *are single-valued and analytic. We confine our attention to such a domain. We assume* Z, X_1, X_2, $\dots$, X_n *to have no common factor.*

Solutions may be obtained by examining the values of z *determined by* $Z=0$, *by* $X_1=0$, *by* $X_2=0, \dots,$ *by* $X_n=0$, *and by seeing whether these satisfy the differential equation* (A). *These solutions are, however, trivial.*

All other solutions are determined as follows:

Let n *functionally independent solutions of the equations*

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\dots=\frac{dx_n}{X_n}=\frac{dz}{Z}$$

be

$$\phi_i(z, x_1, x_2, \dots, x_n)=\phi_i, \qquad i=1, 2, \dots, n.$$

One of the Jacobians

$$J=\frac{\partial(\phi_1, \phi_2, \dots, \phi_n)}{\partial(x_1, x_2, \dots, x_n)}$$

or

$$J_s=\frac{\partial(\phi_1, \phi_2, \dots, \phi_n)}{\partial(z, x_1, x_2, \dots, x_{s-1}, x_{s+1}, \dots, x_n)}, \qquad s=1, 2, \dots, n,$$

does not vanish identically, since $\phi_1, \phi_2, \dots, \phi_n$ *are functionally independent.*

Two cases arise.

CASE I. *If* $J \not\equiv 0$, *the general single-valued analytic solution, except for the trivial solutions afforded by*

$$Z=0,$$

may be written in the form

$$\theta(z)F(\phi_1, \phi_2, \dots, \phi_n)=0,$$

F *being an arbitrary function of* $\phi_1, \phi_2, \dots, \phi_n$ *involving* z, *and* $\theta(z)=0$ *being*

an equation in z alone, determining only those (non-infinite) values of z which satisfy $Z=0$.

CASE II. *If $J_s \not\equiv 0$ (s being an integer such that $1 \leqq s \leqq n$), the general single-valued analytic solution, except for the trivial solutions afforded by*

$$X_s = 0,$$

may be written in the form

$$\theta_s(x_s) F(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

F being an arbitrary function of $\phi_1, \phi_2, \ldots \phi_n$) involving z, and $\theta_s(x_s)$ being a function of x_s alone.

In any case the general single-valued analytic solution of (A), exclusive of the trivial solutions mentioned above, is

$$\theta(z)\theta_1(x_1)\theta_2(x_2)\ldots\theta_n(x_n)F(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

F being an arbitrary function of $\phi_1, \phi_2, \ldots, \phi_n$ which involves z explicitly; and $\theta(z)$, $\theta_1(x_1)$, $\theta_2(x_2)$, $\ldots$, $\theta_n(x_n)$ being functions restricted as before.

As an illustration of the theory let us take the example

$$p_1 \tan x_1 + p_2 \tan x_2 = \tan z.$$

Here we get for our auxiliary solutions

$$\phi_1 = \frac{\sin x_1}{\sin z}, \quad \phi_2 = \frac{\sin x_2}{\sin z}.$$

The Lagrange general solution is then

$$f(\phi_1, \phi_2) = 0.$$

where f is an arbitrary function of ϕ_1 and ϕ_2 involving z explicitly.

We can easily verify that

$$\psi = \sin z - \frac{\sin^{n+1} x_1}{\sin^n x_2} = 0$$

affords a solution of the original equation. But, since

$$\frac{\partial(\phi_1, \phi_2, \psi)}{\partial(z, x_1, x_2)} \not\equiv 0,$$

ψ cannot be put in the form of the Lagrange general integral.

Since $J = \dfrac{\partial(\phi_1, \phi_2)}{\partial(x_1, x_2)} \not\equiv 0$, we know that ψ can be put in the form

$$\theta(z) F(\phi_1, \phi_2);$$

thus we have

$$\psi \equiv \sin z \left(\frac{\phi_2^n - \phi_1^{n+1}}{\phi_2^n} \right).$$

We see that in this example $\theta(z) = \sin z = 0$ determines only those values of z which cause $Z, \equiv \tan z$, to vanish.

Likewise, since

$$J_1 = \frac{\partial(\phi_1, \phi_2)}{\partial(z, x_2)} \not\equiv 0 \quad \text{and} \quad J_2 = \frac{\partial(\phi_1, \phi_2)}{\partial(z, x_1)} \not\equiv 0,$$

ψ can be put in the forms $\theta_1(x_1)F_1(\phi_1, \phi_2)$ and $\theta_2(x_2)F_2(\phi_1, \phi_2)$; thus we have

$$\psi \equiv \sin x_1\left(\frac{1}{\phi_1} - \frac{\phi_1^n}{\phi_2^n}\right) \equiv \sin x_2\left(\frac{1}{\phi_2} \frac{\phi_1^{n+1}}{\phi_2^{n+1}}\right).$$

An example in which $\theta = 0$ or $\theta_i = 0$ may determine an infinite value of the variable is afforded by the equation

$$x_1^2 p_1 - x_1 x_2 p_2 = -x_2^2.$$

Here the auxiliary solutions are

$$\phi_1 = x_1 x_2, \quad \phi_2 = z - \frac{x_2^2}{3x_1}.$$

A so-called special solution is

$$\psi = 3zx_1 - x_2^2 = 0,$$
$$= \frac{3}{x_2}\phi_1\phi_2.$$

Here $\theta_2(x_2)$ is $\frac{3}{x_2}$; equating θ_2 to zero, we have $x_2 = \infty$.

An example of the trivial solutions is afforded by the equation

$$(z - e^{x_2})p_1 + p_2 = z.$$

Here the auxiliary solutions are

$$\phi_1 = \frac{e^{x_2}}{z}, \quad \phi_2 = x_1 - z + e^{x_2}.$$

Trivial solutions are

$$\psi_1 = z - e^{x_2} = 0, \quad \text{from } X_1 = 0,$$

and

$$\psi_2 = z = 0, \quad \text{from } Z = 0.$$

Neither ψ_1 nor ψ_2 is a function of ϕ_1 and ϕ_2.

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